Classical and quantum dynamics of a perfect fluid scalar-metric cosmology

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This work is dedicated to the revered memory of Prof. Masoud Alimohammadi

Abstract

We study the classical and quantum models of a Friedmann-Robertson-Walker (FRW) cosmology, coupled to a perfect fluid, in the context of the scalar-metric gravity. Using the Schutz' representation for the perfect fluid, we show that, under a particular gauge choice, it may lead to the identification of a time parameter for the corresponding dynamical system. It is shown that the evolution of the universe based on the classical cosmology represents a late time power law expansion coming from a big-bang singularity in which the scale factor goes to zero while the scalar field blows up. Moreover, this formalism gives rise to a Schrödinger-Wheeler-DeWitt (SWD) equation for the quantum-mechanical description of the model under consideration, the eigenfunctions of which can be used to construct the wave function of the universe. We use the resulting wave function in order to investigate the possibility of the avoidance of classical singularities due to quantum effects by means of the many-worlds and ontological interpretation of quantum cosmology.

1 Introduction

Classical and semiclassical scalar fields play an essential role in unified theories of interactions and also in all branches of the modern cosmological theories [1]. From a cosmological point of view, there is a renewed interest in the scalar-tensor models in which a non-minimal coupling appears between the geometry of space-time and a scalar field [2]. This is because that a number of different scenarios in cosmology such as spatially flat and accelerated expanding universe at the present time [3], inflation [4], dark matter and dark energy [5], and a rich variety of behaviors can be accommodated phenomenologically by scalar fields. Traditionally cosmological models of inflation use a single scalar field with a canonical kinetic term of the form $1/2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ with some particular self-interaction potential $V(\phi)$ like $1/2m^2\phi^2$ or $\lambda\phi^4$, etc. Such a scalar field is often known as minimally coupled to the geometry. On the other hand, the scalar field in what is qualified to be called the scalar-tensor theory is not simply added to the tensor gravitational field, but comes into play through the non-minimal coupling term [6].

In this paper we shall study the classical and quantum time evolution of a flat FRW model with a perfect fluid matter source and a non-linear self-coupling scalar field minimally coupled to gravity. In our model the the scalar field is coupled to the metric through a coupling function $F(\phi)$ in order to obtain a scalar-metric theory which as we shall see leads to satisfactory cosmological results. The classical cosmology of such a formulation is studied in [7] and it is shown that such scalar-metric

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coupling gives a satisfactory description of the weak fields and a possible way to remove the missing matter problem in non-flat cosmologies. Also, a generalized version of the model studied in [7] in classical and quantum cosmology level is considered in [8].

Here, we first consider a gravitational action in the FRW background in which a scalar field is coupled to the metric with a generic form of a $F(\phi)$ function. To make the model simple and solvable, after some steps, we take a polynomial coupling of the form $F(\phi) = \lambda \phi^m$. For the matter source of gravity, we consider a perfect fluid in Schutz' formalism [9]. The advantage of using this formalism in our quantum cosmological model is that, in a natural way, it can offer a time parameter in terms of dynamical variables of the perfect fluid [10]. Indeed, as we shall show, after a canonical transformation the conjugate momentum associated to one of the variables of the fluid appears linearly in the Hamiltonian of the model. Therefore, canonical quantization results in a Schrödinger-Wheeler-DeWitt (SWD) equation, in which this matter variable plays the role of time. In terms of this time parameter, we shall obtain the dynamical behavior of the cosmic scale factor and the scalar field. We show that the evolution of the scale factor represents a late time power law expansion coming from a big-bang singularity. Also, the classical behavior of the scalar field shows a blow up in this regime. Finally, we deal with the quantization of the model, and by computing the expectation values of the scale factor and the scalar field and also their ontological counterparts, we show that the evolution of the universe according to the quantum picture is free of classical singularities.

2 The classical model

In this section we consider a FRW cosmology in which a scalar field which is coupled to the metric. Also, a perfect fluid with which the action of the model is augmented, plays the role of the matter part of the model. In the context of the ADM formalism the action reads (in what follows we work in units where $c = \hbar = 16\pi G = 1$)

$$\mathcal{S} = \int_M d^4x \sqrt{-g}\,[R - F(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}] + 2\int_{\partial M} d^3x \sqrt{h}\,h_{ab}K^{ab} + \int_M d^4x \sqrt{-g}p, \qquad (1)$$

where R is the scalar curvature and $F(\phi)$ is an arbitrary function of the scalar field ϕ. Also, K^{ab} is the extrinsic curvature and h_{ab} is the induced metric over the three-dimensional spatial hypersurface, which is the boundary ∂M of the four-dimensional manifold M. The last term of (1) denotes the matter contribution to the total action where p is the pressure of perfect fluid which is linked to its energy density by the equation of state

$$p = \alpha\rho. \qquad (2)$$

In Schutz' formalism the fluid's four-velocity is expressed in terms of five potentials ϵ, ζ, β, θ and S as [9]

$$U_\nu = \frac{1}{\mu}(\epsilon_{,\nu} + \zeta\beta_{,\nu} + \theta S_{,\nu}), \qquad (3)$$

where μ is the specific enthalpy, the variable S is the specific entropy while the potentials ζ and β are related to torsion and are absent in the FRW models. The variables ϵ and θ have no clear physical interpretation in this formalism. The four-velocity satisfies the condition

$$U^\nu U_\nu = -1. \qquad (4)$$

We assume that the geometry of spacetime is described by the FRW metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right], \qquad (5)$$

where $N(t)$ is the lapse function, $a(t)$ the scale factor and k=1, 0 and -1 correspond to the closed, flat and open universe respectively. To proceed further, we need an effective Lagrangian for the model

whose variation with respect to its dynamical variables yields the appropriate equations of motion. Therefore, by considering the above action as representing a dynamical system in which the scale factor a, scalar field ϕ and fluid's potentials play the role of independent dynamical variables, we can rewrite the gravitational part of the action (1) as

$$\mathcal{S}_{grav} = \int dt \mathcal{L}_{grav}(a, \dot{a}, \phi, \dot{\phi}) = \int dt \left\{ Na^3 \left(R + \frac{1}{N^2}F(\phi)\dot{\phi}^2 \right) - \lambda \left[R - \frac{6}{N^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{N\dot{a}}{Na} \right) \right] \right\}, \quad (6)$$

where we have introduced the definition of R in terms of a and its derivatives as a constraint. This procedure allows us to remove the second order derivatives from action (6). The Lagrange multiplier λ can be obtained by variation with respect to R, that is, $\lambda = Na^3$. Thus, we obtain the following point-like Lagrangian for the gravitational part of the model

$$\mathcal{L}_{grav} = -\frac{6}{N}a\dot{a}^2 + 6kNa + \frac{1}{N}F(\phi)a^3\dot{\phi}^2. \quad (7)$$

Also, the matter part of the action (1) becomes $\mathcal{S}_{matt} = \int d^3x dt Na^3 p$, so the Lagrangian density of the fluid is $\mathcal{L}_{matt} = Na^3 p$. Following the thermodynamic description of [9], the basic thermodynamic relations take the form

$$\rho = \rho_0(1 + \Pi), \quad \mu = 1 + \Pi + \frac{p}{\rho_0}, \quad (8)$$

where ρ_0 and Π are the rest-mass density and the specific internal energy of the fluid respectively. These quantities together with the temperature of the system τ, obey the first law of the thermodynamics $\tau dS = d\Pi + pd(1/\rho_0)$, which can be rewritten as

$$\tau dS = d\Pi + pd \left(\frac{1}{\rho_0} \right) = (1 + \Pi)d \left[\ln \rho_0^{-\alpha}(1 + \Pi) \right], \quad (9)$$

where we have used the equation of state (2). Therefore, we obtain the following expressions for the temperature and the entropy of the fluid

$$\tau = 1 + \Pi, \quad S = \ln \rho_0^{-\alpha}(1 + \Pi) = \ln \frac{\mu}{\alpha + 1}\rho_0^{-\alpha}. \quad (10)$$

Now, we can express ρ_0 and Π as functions of μ and S as

$$1 + \Pi = \frac{\mu}{\alpha + 1}, \quad \rho_0 = \left(\frac{\mu}{\alpha + 1} \right)^{1/\alpha} e^{-S/\alpha}, \quad (11)$$

so that with the help of (8), one can put the equation of state in the form

$$p = \frac{\alpha}{(\alpha + 1)^{1+1/\alpha}}\mu^{1+1/\alpha}e^{-S/\alpha}. \quad (12)$$

On the other hand, normalization of the fluid's four-velocity (3), according to the relation (4) implies $\mu = (\dot{\epsilon} + \theta\dot{S})/N$. Therefore, using the above constraints and thermodynamical considerations for the fluid we find

$$\mathcal{L}_{matt} = N^{-1/\alpha}a^3 \frac{\alpha}{(\alpha + 1)^{1+1/\alpha}} \left(\dot{\epsilon} + \theta\dot{S} \right)^{1+1/\alpha} e^{-S/\alpha}. \quad (13)$$

Let us now construct the Hamiltonian for our model. The momenta conjugate to each of the above variables can be obtained from the definition $P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. In terms of the conjugate momenta the Hamiltonian is given by

$$H = H_{grav} + H_{matt} = \dot{a}P_a + \dot{\phi}P_\phi + \dot{\epsilon}P_\epsilon + \dot{S}P_S - \mathcal{L}, \quad (14)$$

where $\mathcal{L} = \mathcal{L}_{grav} + \mathcal{L}_{matt}$. Expression (14) leads to

$$H = N\mathcal{H} = N \left[-\frac{1}{24}\frac{P_a^2}{a} + \frac{1}{4F(\phi)a^3}P_\phi^2 - 6ka + a^{-3\alpha}e^S P_\epsilon^{\alpha+1} \right]. \quad (15)$$

Now, consider the following canonical transformation which is a generalization of the ones used in [11]

$$T = -P_S e^{-S} P_\epsilon^{-(\alpha+1)}, \quad P_T = P_\epsilon^{\alpha+1} e^S,$$

$$\bar{\epsilon} = \epsilon - (\alpha + 1)\frac{P_S}{P_\epsilon}, \qquad \bar{P}_\epsilon = P_\epsilon.$$
(16)

Under this transformation Hamiltonian (15) takes the form

$$H = N\mathcal{H} = N\left[-\frac{1}{24}\frac{P_a^2}{a} + \frac{1}{4F(\phi)a^3}P_\phi^2 - 6ka + \frac{P_T}{a^{3\alpha}}\right].$$
(17)

We see that the momentum P_T is the only remaining canonical variable associated with the matter and appears linearly in the Hamiltonian. The setup for constructing the phase space and writing the Lagrangian and Hamiltonian of the model is now complete.

The classical dynamics is governed by the Hamiltonian equations, that is

$$\begin{cases} \dot{a} = \{a, H\} = -\frac{N}{12}\frac{P_a}{a}, \\[2mm] \dot{P}_a = \{P_a, H\} = N\left[-\frac{1}{24}\frac{P_a^2}{a^2} + \frac{3}{4F(\phi)a^4}P_\phi^2 + 6k + 3\alpha a^{-3\alpha-1}P_T\right], \\[2mm] \dot{\phi} = \{\phi, H\} = \frac{N}{2F(\phi)a^3}P_\phi, \\[2mm] \dot{P}_\phi = \{P_\phi, H\} = N\frac{P_\phi^2}{4a^3}\frac{F'(\phi)}{F(\phi)^2}, \\[2mm] \dot{T} = \{T, H\} = \frac{N}{a^{3\alpha}}, \\[2mm] \dot{P}_T = \{P_T, H\} = 0. \end{cases}$$
(18)

We also have the constraint equation $\mathcal{H} = 0$. Up to this point the cosmological model, in view of the concerning issue of time, has been of course under-determined. Before trying to solve these equations we must decide on a choice of time in the theory. The under-determinacy problem at the classical level may be resolved by using the gauge freedom via fixing the gauge. A glance at the above equations shows that choosing the gauge $N = a^{3\alpha}$, we have

$$N = a^{3\alpha} \Rightarrow T = t,$$
(19)

which means that variable T may play the role of time in the model. Therefore, the classical equations of motion can be rewritten in the gauge $N = a^{3\alpha}$ as follows

$$\begin{cases} \dot{a} = -\frac{1}{12}a^{3\alpha-1}P_a, \\[2mm] \dot{P}_a = -\frac{1}{24}a^{3\alpha-2}P_a^2 + \frac{3}{4F(\phi)}a^{3\alpha-4}P_\phi^2 + 6ka^{3\alpha} + 3\alpha P_0 a^{-1}, \\[2mm] \dot{\phi} = \frac{P_\phi}{2F(\phi)}a^{3\alpha-3}, \\[2mm] \dot{P}_\phi = \frac{1}{4}\frac{F'(\phi)}{F(\phi)^2}P_\phi^2 a^{3\alpha-3}, \end{cases}$$
(20)

where we take $P_T = P_0 = $ const. from the last equation of (18). The two last equations of the above system indicate that the field ϕ obeys the second order equation of motion

$$\frac{\ddot{\phi}}{\dot{\phi}} + \frac{1}{2}\frac{F'(\phi)}{F(\phi)}\dot{\phi} + 3(1-\alpha)\frac{\dot{a}}{a} = 0,$$
(21)

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which is equivalent to the conservation law, provided the standard perfect fluid energy-momentum tensor is introduced. Equation (21) can easily be integrated to yield

$$\dot{\phi}^2 F(\phi) = Ca^{6(\alpha-1)}, \tag{22}$$

where C is an integration constant. Also, eliminating the momenta from the system (20) results

$$-6a^{1-3\alpha}\dot{a}^2 + \dot{\phi}^2 F(\phi)a^{3-3\alpha} - 6ka^{1+3\alpha} + P_0 = 0, \tag{23}$$

which is nothing but the constraint equation $\mathcal{H} = 0$. With the help of (22) this equation can be put into the form

$$6\dot{a}^2 = Ca^{6\alpha-4} - 6ka^{6\alpha} + P_0a^{3\alpha-1}, \tag{24}$$

where for the flat case $k = 0$, its solution reads

$$a(t) = \left[\frac{3P_0(1-\alpha)^2}{8}(t-\delta)^2 - \frac{C}{P_0}\right]^{\frac{1}{3(1-\alpha)}}, \tag{25}$$

for $\alpha \neq 1$, and

$$a(t) = a_0 \exp\left(\sqrt{\frac{C+P_0}{6}}t\right), \tag{26}$$

for $\alpha = 1$, with constants δ and a_0. We may set $\delta = \frac{\sqrt{8C}}{\sqrt{3}P_0(1-\alpha)}$, so that $a(t=0) = 0$. What remains to be found is an expression for the scalar field $\phi(t)$. In the following, we shall consider the case of a coupling function in the form $F(\phi) = \lambda\phi^m$. With this choice for the function $F(\phi)$ we are able to calculate the time evolution of the scalar field as

$$\phi(t) = \begin{cases} \left\{\frac{\sqrt{2}(m+2)}{\sqrt{3\lambda}(1-\alpha)}\tanh^{-1}\left[\frac{\sqrt{3}P_0(1-\alpha)}{\sqrt{8C}}(t-\delta)\right]\right\}^{\frac{2}{m+2}}, & m \neq -2, \\ \exp\left\{\frac{2\sqrt{2}}{\sqrt{3\lambda}(1-\alpha)}\tanh^{-1}\left[\frac{\sqrt{3}P_0(1-\alpha)}{\sqrt{8C}}(t-\delta)\right]\right\}, & m = -2, \end{cases} \tag{27}$$

for $\alpha \neq 1$ and

$$\phi(t) = \begin{cases} \left[\sqrt{\frac{C}{\lambda}}\frac{m+2}{2}(t-\delta)\right]^{\frac{2}{m+2}}, & m \neq -2, \\ \phi_0 e^{\sqrt{\frac{C}{\lambda}}(t-\delta)}, & m = -2, \end{cases} \tag{28}$$

for $\alpha = 1$. We see that for $\alpha \neq 1$, the evolution of the universe based on (25) begins with a big-bang singularity at $t = 0$ and, for $\alpha < 1$, follows the power law expansion $a(t) \sim t^{\frac{2}{3(1-\alpha)}}$ at late time of cosmic evolution while the scalar field has a monotonically increasing behavior coming from $\phi \to -\infty$, reaches zero and then blows up at a finite time. Also, in the case of $\alpha = 1$ (stiff matter), an exponential solution is obtained for the corresponding cosmology in the chosen time variable. In terms of cosmic time $\eta = \int Ndt$, solution (26) reads as $a(\eta) \sim \eta^{1/3}$ which shows a decelerated expansion. This not surprising since for $\alpha = 1$ there is no violation of the strong energy condition, and hence an accelerated expansion cannot be obtained. To understand the relation between the big-bang singularity $a \to 0$ and the blow up singularity $\phi \to \pm\infty$, we are going to find a classical trajectory in configuration space (a, ϕ), where the time parameter t is eliminated. From (22) and (24) one has

$$\frac{\sqrt{F(\phi)}d\phi}{da} = \pm\frac{\sqrt{C}a^{3(\alpha-1)}}{\sqrt{\frac{C}{6}a^{6\alpha-4} - ka^{6\alpha} + \frac{P_0}{6}a^{3\alpha-1}}}, \tag{29}$$

where for the case $k = 0$, after integration reads

$$\phi(a) = \left\{ \pm \frac{\sqrt{2}(m+2)}{\sqrt{3\lambda}(1-\alpha)} \sinh^{-1}\left[\sqrt{\frac{C}{P_0}} a^{\frac{3(\alpha-1)}{2}} \right] \right\}^{\frac{2}{m+2}}. \tag{30}$$

Equation (30) describes two branches for which $\phi \to 0$ (or a non-zero constant if we add a non-zero integration constant to the above relation) if $a \to \infty$ and $\phi \to \pm\infty$ if $a \to 0$. This means that at the late time limit, the scalar field approaches a constant value while its blow up behavior corresponds to the big-bang singularity. We shall see in the next section that this classical picture will be modified if one takes into account the quantum mechanical considerations in the problem at hand.

3 The quantum model

We now focus attention on the study of the quantum cosmology of the model described above. We start by writing the Wheeler-DeWitt equation from Hamiltonian (17). A remark about our quantization procedure is that the canonical transformation (16) is applied to the classical Hamiltonian (15), resulting in Hamiltonian (17) which we are going to quantize. To make this acceptable, one should show that in the quantum theory the two Hamiltonians are connected by some unitary transformation, i.e. the transformation (16) is also a quantum canonical transformation. A quantum canonical transformation is defined as a change of the phase space variables $(q, p) \to (q', p')$ which preserves the Dirac bracket [12]

$$[q, p] = i = [q'(q, p), p'(q, p)]. \tag{31}$$

Such a transformation is implemented by a function $C(q, p)$ such that

$$q'(q, p) = CqC^{-1}, \quad p'(q, p) = CpC^{-1}. \tag{32}$$

This canonical transformation C, transforms the Hamiltonian as $H'(q, p) = CH(q, p)C^{-1}$. For our case the canonical relations $[S, P_S] = [\epsilon, P_\epsilon] = i$ yield

$$[T, P_T] = \left[-P_S e^{-S} P_\epsilon^{-(\alpha+1)}, P_\epsilon^{\alpha+1} e^S \right] = - \left[P_S e^{-S}, e^S \right] = \left[e^S, P_S \right] e^{-S} = i e^S e^{-S} = i, \tag{33}$$

and

$$[\bar{\epsilon}, \bar{P}_\epsilon] = \left[\epsilon - (\alpha+1) P_S P_\epsilon^{-1}, P_\epsilon \right] = [\epsilon, P_\epsilon] = i, \tag{34}$$

which means that the transformation (16) preserves the Dirac brackets and thus is a quantum canonical transformation. Therefore, use of the transformed Hamiltonian (17) for quantization of the model is quite reasonable.

3.1 Schrödinger-Wheeler-DeWitt equation

Since the lapse function N appears as a Lagrange multiplier in the Hamiltonian (17), we have the Hamiltonian constraint $\mathcal{H} = 0$. Thus, application of the Dirac quantization procedure demands that the quantum states of the universe should be annihilated by the operator version of $\mathcal{H}$, that is

$$\mathcal{H}\Psi(a, \phi, T) = \left[-\frac{1}{24}\frac{P_a^2}{a} + \frac{1}{4F(\phi)a^3}P_\phi^2 - 6ka + \frac{P_T}{a^{3\alpha}} \right] \Psi(a, \phi, T) = 0, \tag{35}$$

where $\Psi(a, \phi, T)$ is the wave function of the universe. Choice of the ordering $a^{-1}P_a^2 = P_a a^{-1}P_a$ and $F(\phi)^{-1}P_\phi^2 = P_\phi F(\phi)^{-1}P_\phi$ to make the Hamiltonian Hermitian and use of the usual representation $P_q \to -i\partial_q$ results in

$$\left[a^{-1}\frac{\partial^2}{\partial a^2} - a^{-2}\frac{\partial}{\partial a} - 6a^{-3}F(\phi)^{-1}\frac{\partial^2}{\partial \phi^2} + 6a^{-3}\frac{F'(\phi)}{F(\phi)^2}\frac{\partial}{\partial \phi} - 24ia^{-3\alpha}\frac{\partial}{\partial T} \right] \Psi(a, \phi, T) = 0. \tag{36}$$

This equation takes the form of a Schrödinger equation $i\partial\Psi/\partial T = H\Psi$, in which the Hamiltonian operator is Hermitian with the standard inner product

$$< \Phi \mid \Psi >= \int_{(a,\phi)} a^{-3\alpha}\Phi^*\Psi da d\phi. \tag{37}$$

We separate the variables in the SWD equation (36) as

$$\Psi(a,\phi,T) = e^{iET}\psi(a,\phi), \tag{38}$$

leading to

$$\left[a^2\frac{\partial^2}{\partial a^2} - a\frac{\partial}{\partial a} - \frac{6}{F(\phi)}\frac{\partial^2}{\partial\phi^2} + 6\frac{F'(\phi)}{F(\phi)^2}\frac{\partial}{\partial\phi} - 24Ea^{3-3\alpha}\right]\psi(a,\phi) = 0, \tag{39}$$

where E is a separation constant. The solutions of the above differential equation are separable and may be written in the form $\psi(a,\phi) = U(a)V(\phi)$ which yields

$$\begin{cases} \left[a^2\frac{d^2}{da^2} - a\frac{d}{da} + (24Ea^{3-3\alpha} + \nu^2)\right]U(a) = 0, \\ \\ \left[\frac{6}{F(\phi)}\frac{d^2}{d\phi^2} - 6\frac{F'(\phi)}{F(\phi)^2}\frac{d}{d\phi} + \nu^2\right]V(\phi) = 0, \end{cases} \tag{40}$$

where ν is another constant of separation. Upon substituting the relation $F(\phi) = \lambda\phi^m$ into the above system, its solutions read in terms of Bessel functions J and Y as

$$\begin{cases} U(a) = a\left[c_1 J_{\frac{2\sqrt{1-\nu^2}}{3(1-\alpha)}}\left(\frac{\sqrt{96E}}{3(1-\alpha)}a^{\frac{3(1-\alpha)}{2}}\right) + c_2 Y_{\frac{2\sqrt{1-\nu^2}}{3(1-\alpha)}}\left(\frac{\sqrt{96E}}{3(1-\alpha)}a^{\frac{3(1-\alpha)}{2}}\right)\right], \\ \\ V(\phi) = \phi^{\frac{m+1}{2}}\left[d_1 J_{\frac{m+1}{m+2}}\left(\frac{\nu\sqrt{6\lambda}}{3(m+2)}\phi^{\frac{m+2}{2}}\right) + d_2 Y_{\frac{m+1}{m+2}}\left(\frac{\nu\sqrt{6\lambda}}{3(m+2)}\phi^{\frac{m+2}{2}}\right)\right], \end{cases} \tag{41}$$

where $c_i(i = 1,2)$ and $d_i(i = 1,2)$ are integration constants. Thus, the eigenfunctions of the SWD equation can be written as

$$\Psi_{\nu E}(a,\phi,T) = e^{iET}a\phi^{\frac{m+1}{2}}J_{\frac{2\sqrt{1-\nu^2}}{3(1-\alpha)}}\left(\frac{\sqrt{96E}}{3(1-\alpha)}a^{\frac{3(1-\alpha)}{2}}\right)J_{\frac{m+1}{m+2}}\left(\frac{\nu\sqrt{6\lambda}}{3(m+2)}\phi^{\frac{m+2}{2}}\right), \tag{42}$$

where we have chosen $c_2 = d_2 = 0$ for having well-defined functions in all ranges of variables a and ϕ. We may now write the general solutions to the SWD equations as a superposition of the eigenfunctions, that is

$$\Psi(x,y,T) = \int_{E=0}^{\infty}\int_{\nu=0}^{1} A(E)C(\nu)\Psi_{\nu E}(x,y,T)dEd\nu, \tag{43}$$

where $A(E)$ and $C(\nu)$ are suitable weight functions to construct the wave packets. By using the equality [13]

$$\int_0^{\infty} e^{-ar^2}r^{\nu+1}J_{\nu}(br)dr = \frac{b^{\nu}}{(2a)^{\nu+1}}e^{-\frac{b^2}{4a}}, \tag{44}$$

we can evaluate the integral over E in (43) and simple analytical expression for this integral is found if we choose the function $A(E)$ to be a quasi-Gaussian weight factor

$$A(E) = \frac{16}{3(1-\alpha)^2}\left(\frac{\sqrt{96E}}{3(1-\alpha)}\right)^{\sqrt{1-\nu^2}}\exp\left(-\frac{32\gamma}{3(1-\alpha)^2}E\right), \tag{45}$$

which results in

$$\int_0^\infty A(E)e^{iET}J_{\frac{2\sqrt{1-\nu^2}}{3(1-\alpha)}}\left(\frac{\sqrt{96E}}{3(1-\alpha)}a^{\frac{3(1-\alpha)}{2}}\right)dE \;\; =$$

$$a^{\sqrt{1-\nu^2}}\left[2\gamma - i\frac{3}{16}(1-\alpha)^2T\right]^{-1-\frac{2\sqrt{1-\nu^2}}{3(1-\alpha)}}\exp\left[-\frac{a^{3(1-\alpha)}}{4\gamma - i\frac{3}{8}(1-\alpha)^2T}\right], \qquad (46)$$

where γ is an arbitrary positive constant. Substitution of the above relation into equation (43) leads to the following expression for the wave function

$$\Psi(a,\phi,T) \;\; = \;\; a\left[2\gamma - i\frac{3}{16}(1-\alpha)^2T\right]^{-1}\phi^{\frac{m+1}{2}}\exp\left[-\frac{a^{3(1-\alpha)}}{4\gamma - i\frac{3}{8}(1-\alpha)^2T}\right]$$

$$\times \int_0^1 C(\nu)a^{\sqrt{1-\nu^2}}\left[2\gamma - i\frac{3}{16}(1-\alpha)^2T\right]^{-\frac{2\sqrt{1-\nu^2}}{3(1-\alpha)}}J_{\frac{m+1}{m+2}}\left(\frac{\nu\sqrt{6\lambda}}{3(m+2)}\phi^{\frac{m+2}{2}}\right)d\nu. \quad (47)$$

To achieve an analytical closed expression for the wave function, we assume that the above superposition is taken over such values of ν for which one can use the approximation $\sqrt{1-\nu^2}\sim 1$. Now, by using the equality [13]

$$\int_0^1 \nu^{r+1}(1-\nu^2)^{s/2}J_r(z\nu)d\nu = \frac{2^s\Gamma(s+1)}{z^{s+1}}J_{r+s+1}(z), \qquad (48)$$

and choosing the weight function $C(\nu) = \nu^{\frac{2m+3}{m+2}}(1-\nu^2)^{s/2}$, we are led to the following expression for the wave function [1]

$$\Psi(a,\phi,T) \;\; = \;\; \mathcal{N}a^2\left[2\gamma - i\frac{3}{16}(1-\alpha)^2T\right]^{\frac{3\alpha-5}{3(1-\alpha)}}\phi^{-\frac{s(m+2)+1}{2}}\exp\left[-\frac{a^{3(1-\alpha)}}{4\gamma - i\frac{3}{8}(1-\alpha)^2T}\right]$$

$$\times J_{\frac{2m+3}{m+2}+s}\left(\frac{\sqrt{6\lambda}}{3(m+2)}\phi^{\frac{m+2}{2}}\right), \qquad (49)$$

where s and $\mathcal{N}$ are an arbitrary constant and a numerical factor respectively. Now, having the above expression for the wave function of the universe, we are going to obtain the predictions for the behavior of the dynamical variables in the corresponding cosmological model. To do this, in the next subsection, we shall adopt two approaches to evaluate the classical behavior of the dynamical variables in the model which lead to the same results. In the many-worlds interpretation of quantum mechanics [14], we can calculate the expectation values of the dynamical variables and, in the realm of the ontological interpretation of quantum mechanics [16], one can evaluate the Bohmian trajectories for those variables. In figure 1 we have plotted the square of the wave function for typical numerical values of the parameters. As this figure shows, at $T=0$, the wave function has two dominant peaks in the vicinity of some non-zero values of a and ϕ. This means that the wave function predicts the emergence of the universe from a state corresponding to one of its dominant peaks. However, the emergence of several peaks in the wave packet may be interpreted as a representation of different quantum states that may communicate with each other through tunneling. This means that there are different possible universes (states) from which our present universe could have evolved and tunneled in the past, from one universe (state) to another. As time progresses, the wave packet begins to propagate in the a-direction, its width becoming wider and its peaks moving with a group velocity towards the greater values of a while the values of ϕ remaining almost constant. The wave packet

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[1]One may have some doubts on this final form for the wave function and the following results due to the assumption $\sqrt{1-\nu^2}\sim 1$. Indeed, since the wave function can be a complex function, we may extend the integration domain over ν to $0\le\nu<\infty$. In this case with a numerical study of equation (47), we have verified that the general patterns of the resulting wave packets follow the behavior shown in figure 1 with a very good approximation.
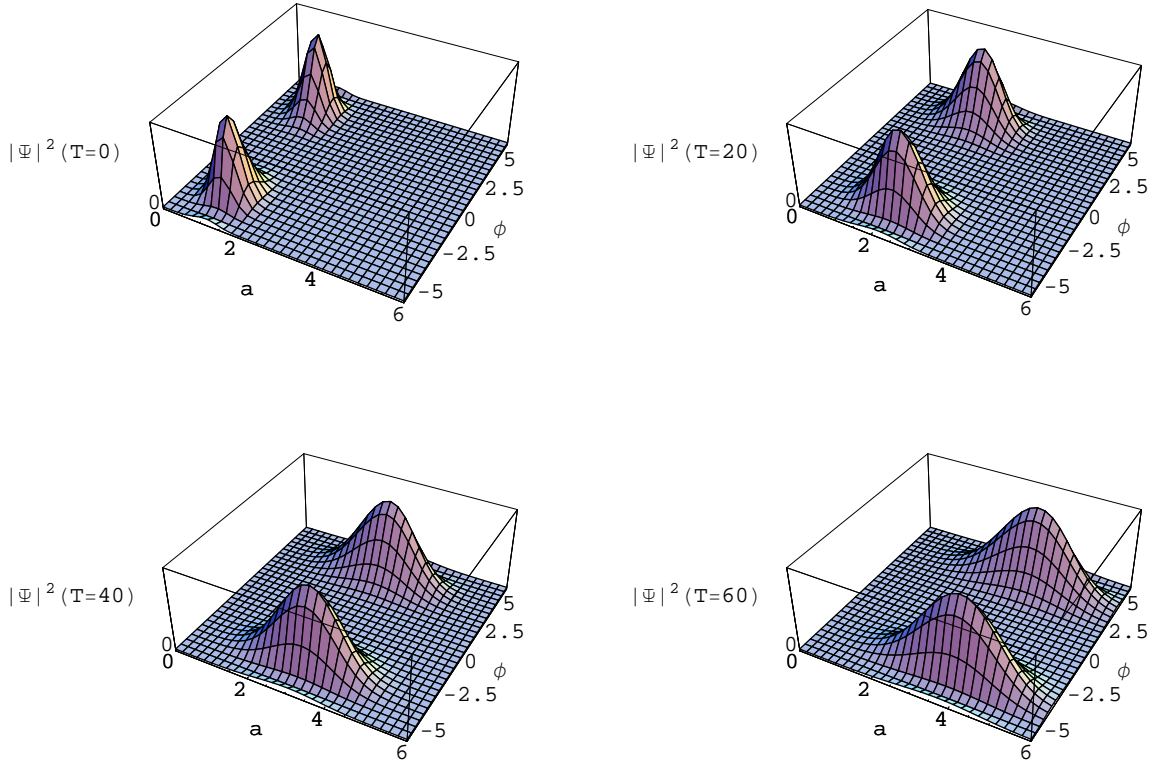
Figure 1: The figures show $|\Psi(a, \phi, T)|^2$, the square of the wave function in four different time parameter $T = 0, 20, 40, 60$. The figures are plotted for the numerical values $\gamma = 1$, $m = 2$, $s = 1$, $\lambda = 1$, and we set the equation of state parameter $\alpha = -1/3$. After examining some other values for this parameter, we verify that the general behavior of the wave function is repeated.

disperses as time passes, the minimum width being attained at $T = 0$. As in the case of the free particle in quantum mechanics, the more localized the initial state at $T = 0$, the more rapidly the wave packet disperses. Therefore, the quantum effects make themselves felt only for small enough T corresponding to small a, as expected and the wave function predicts that the universe will assume states with larger a and an almost constant ϕ in its late time evolution.

3.2 Recovery of the classical solutions

In general, one of the most important features in quantum cosmology is the recovery of classical cosmology from the corresponding quantum model or, in other words, how can the WD wave function s predict a classical universe. In this approach, one usually constructs a coherent wavepacket with good asymptotic behavior in the minisuperspace, peaking in the vicinity of the classical trajectory. On the other hand, in an another approach to show the correlations between classical and quantum pattern, following the many-worlds interpretation of quantum mechanics [14], one may calculate the time dependence of the expectation value of a dynamical variable q as

$$< q > (t) = \frac{< \Psi|q|\Psi >}{< \Psi|\Psi >}. \tag{50}$$

Following this approach, we may write the expectation value for the scale factor as

$$< a > (T) = \frac{\int_{a=0}^{\infty} \int_{\phi=-\infty}^{\infty} a^{-3\alpha} \Psi^* a \Psi da d\phi}{\int_{a=0}^{\infty} \int_{\phi=-\infty}^{\infty} a^{-3\alpha} \Psi^* \Psi da d\phi}, \tag{51}$$

which yields

$$< a > (T) = \frac{\Gamma(\frac{2-\alpha}{1-\alpha})}{\Gamma(\frac{5-3\alpha}{3-3\alpha})} \left[2\gamma + \frac{9}{512\gamma}(1-\alpha)^4 T^2 \right]^{\frac{1}{3(1-\alpha)}}. \tag{52}$$

It is important to classify the nature of the quantum model as concerns the presence or absence of singularities. For the wave function (49), the expectation value (52) of a never vanishes, showing that these states are nonsingular. Indeed, the expression (52) for $\alpha < 1$, represents a bouncing universe with no singularity where its late time behavior coincides to the late time behavior of the classical solution (25), that is $a(t) \sim t^{\frac{2}{3(1-\alpha)}}$. Now we can calculate the dispersion of the wave packet in the a-direction which is defined as

$$(\Delta a)^2 = < a^2 > - < a >^2, \tag{53}$$

using (49) and (52), we get

$$(\Delta a)^2 \sim \left[2\gamma + \frac{9}{512\gamma}(1-\alpha)^4 T^2 \right]^{\frac{2}{3(1-\alpha)}}. \tag{54}$$

The result is that the wave packet traveling in the a-direction, spreads as time increases and thus its degree of localization is reduced. The width of the wave packet evaluated in (54) agree with the discussion in the end of the previous subsection. Indeed, we may interpret the above relation for the width of the wave function as the coincidence of the classical trajectories with the quantum ones for large values of time. Also, the expectation value for the scalar field reads as

$$< \phi > (T) = \frac{\int_{a=0}^{\infty} \int_{\phi=-\infty}^{\infty} a^{-3\alpha} \Psi^* \phi \Psi da d\phi}{\int_{a=0}^{\infty} \int_{\phi=-\infty}^{\infty} a^{-3\alpha} \Psi^* \Psi da d\phi}, \tag{55}$$

with the result

$$< \phi > (T) = \frac{\int \phi^{-s(m+2)} \left[J_{\frac{2m+3}{m+2}} \left(\frac{\sqrt{6\lambda}}{3(m+2)} \phi^{\frac{m+2}{2}} \right) \right]^2 d\phi}{\int \phi^{-s(m+2)-1} \left[J_{\frac{2m+3}{m+2}} \left(\frac{\sqrt{6\lambda}}{3(m+2)} \phi^{\frac{m+2}{2}} \right) \right]^2 d\phi} = \text{const.}. \tag{56}$$

We see that the expectation value of ϕ does not depend on time which is just the behavior predicted by the wave function of the SWD equation. This result is comparable with those obtained in [15] where a constant expectation value for the dilatonic field in a quantum cosmological model based on the string effective action coupled to matter has been obtained. From the classical solutions in the previous section, it is clear that this is the classical trajectory obtained from (30) in the limit $a \to \infty$. Therefore, in view of the behavior of the scale factor and the scalar field, the classical solutions (52) and (56) are in complete agreement with the quantum patterns shown in figure 1, and both predict a (nonsingular) monotonically increasing evolution for the scale factor and consequently there is an almost good correlation between the quantum patterns and classical trajectories.

The issue of the correlation between classical and quantum schemes may be addressed from another point of view. It is known that the results obtained by using the many-world interpretation agree with those that can be obtained using the ontological interpretation of quantum mechanics [16]. In Bohmian interpretation, the wave function is written as

$$\Psi(a, \phi, T) = \Omega(a, \phi, T)e^{iS(a, \phi, T)}, \tag{57}$$

where Ω and S are some real functions. Substitution of this expression into the SWD equation (36) leads to the continuity equation

$$2a^2\frac{\partial\Omega}{\partial a}\frac{\partial S}{\partial a} + a^2\Omega\frac{\partial^2 S}{\partial a^2} + a\Omega\frac{\partial S}{\partial a} - \frac{12}{F(\phi)}\frac{\partial\Omega}{\partial\phi}\frac{\partial S}{\partial\phi} - \frac{6}{F(\phi)}\Omega\frac{\partial^2 S}{\partial\phi^2} + 6\frac{F'(\phi)}{F(\phi)^2}\Omega\frac{\partial S}{\partial\phi} - 24a^{3(1-\alpha)}\frac{\partial\Omega}{\partial T} = 0, \tag{58}$$

and the modified Hamilton-Jacobi equation

$$-\frac{1}{24}\frac{1}{a}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{4F(\phi)a^3}\left(\frac{\partial S}{\partial\phi}\right)^2 + a^{-3\alpha}\left(\frac{\partial S}{\partial T}\right) + \mathcal{Q} = 0, \tag{59}$$

in which the quantum potential $\mathcal{Q}$ is defined as

$$\mathcal{Q} = \frac{1}{24a\Omega}\frac{\partial^2\Omega}{\partial a^2} + \frac{1}{24a^2\Omega}\frac{\partial\Omega}{\partial a} - \frac{1}{4a^3F(\phi)\Omega}\frac{\partial^2\Omega}{\partial\phi^2} + \frac{F'(\phi)}{4a^3F(\phi)^2\Omega}\frac{\partial\Omega}{\partial\phi}. \tag{60}$$

The real functions $\Omega(a, \phi, T)$ and $S(a, \phi, T)$ can be obtained from the wave function (49) as

$$\Omega = \frac{a^2\phi^{-\frac{s(m+2)+1}{2}}}{\left[4\gamma^2 + \frac{9}{256}(1-\alpha)^4 T^2\right]^{\frac{5-3\alpha}{6(1-\alpha)}}} \exp\left[-\frac{4\gamma a^{3(1-\alpha)}}{16\gamma^2 + \frac{9}{64}(1-\alpha)^4 T^2}\right] J_{\frac{2m+3}{m+2}+s}\left(\frac{\sqrt{6\lambda}}{3(m+2)}\phi^{\frac{m+2}{2}}\right), \tag{61}$$

$$S = -\frac{3}{8}\frac{a^{3(1-\alpha)}(1-\alpha)^2 T}{16\gamma^2 + \frac{9}{64}(1-\alpha)^4 T^2} + \frac{5-3\alpha}{3(1-\alpha)}\arctan\left[\frac{3(1-\alpha)^2 T}{32\gamma}\right]. \tag{62}$$

In this interpretation the classical trajectories, which determine the behavior of the scale factor and scalar field are given by

$$P_a = \frac{\partial S}{\partial a}, \quad P_\phi = \frac{\partial S}{\partial\phi}. \tag{63}$$

Using the expressions for P_a and P_ϕ in (20), the equations for the classical trajectories become

$$-12a^{1-3\alpha}\dot{a} = -\frac{9}{8}\frac{(1-\alpha)^3 T a^{2-3\alpha}}{16\gamma^2 + \frac{9}{64}(1-\alpha)^4 T^2}, \tag{64}$$

and

$$2F(\phi)a^{3(1-\alpha)}\dot{\phi} = 0. \tag{65}$$

Therefore, after integration we get

$$a(T) = a_0\left[16\gamma^2 + \frac{9}{64}(1-\alpha)^4 T^2\right]^{\frac{1}{3(1-\alpha)}}, \tag{66}$$
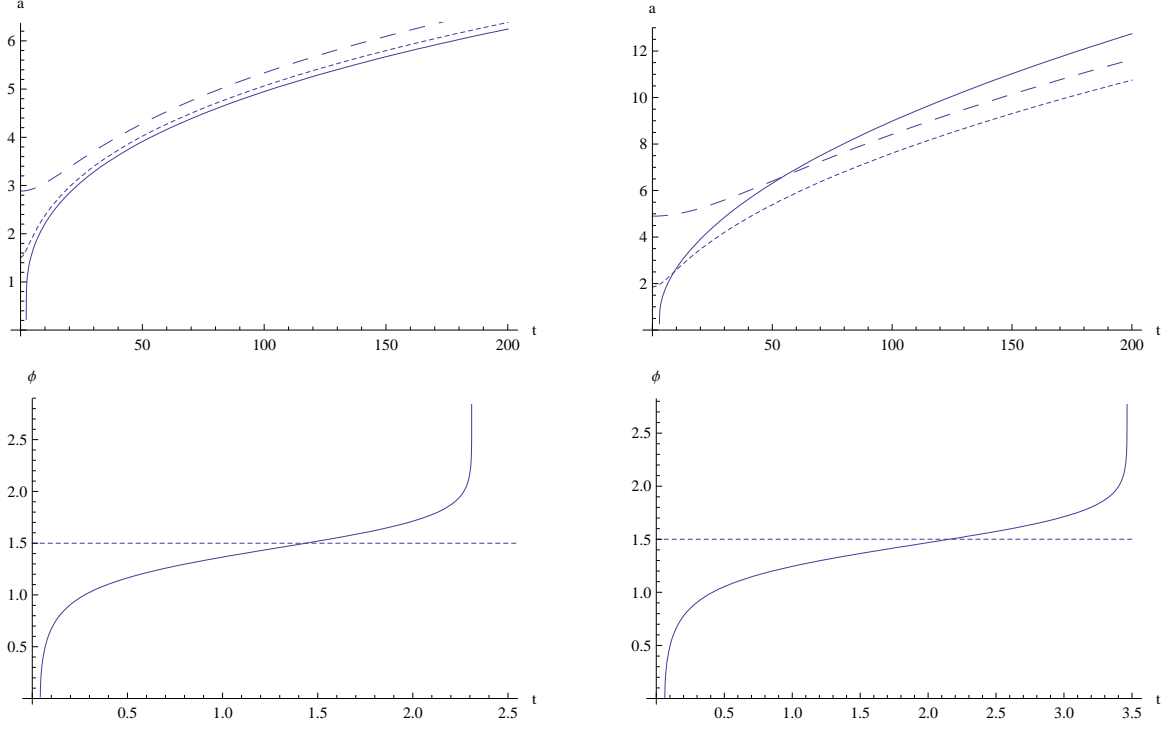
11

Figure 2: Up: the figures show the classical scale factor (solid line), the expectation value of the scale factor (small-dashed line) and the Bohmian scale factor (large-dashed line). Down: the classical scalar field (solid line) and the expectation value or its Bohmian version (dashed line) versus time. The figures are plotted for the numerical values $\gamma = 6$, $P_0 = 1$, $C = 2$, $a_0 = 1$, and we set the equation of state parameters $\alpha = -1$ (left) and $\alpha = -1/3$ (right). After examining some other values for this parameter, we verify that the general behavior of the curves is repeated.

and

$$\phi(T) = \text{const.}, \tag{67}$$

where a_0 is a constant of integration. These solutions have the same behavior as the expectation values computed in (52) and (56) and like those are free of singularity. Figure 2 shows the behavior of the classical scale factor (25), the quantum mechanical expectation value of the scale factor (52) and its Bohmian version (66) versus time for some typical numerical values of the parameters. Also, the behavior of the classical scalar field (27) and its quantum mechanical expectation value is shown in this figure. The origin of the singularity avoidance may be understood by the existence of the quantum potential which corrects the classical equations of motion. To get an approximate scheme of this issue let us neglect the ϕ-terms in (60) because of the constant value for ϕ from (67). Now, inserting the relation (66) in (61), we can find the quantum potential in terms of the scale factor as

$$\mathcal{Q}(a) = \frac{(3\alpha - 1)(3\alpha - 2)}{48} a^{-3}. \tag{68}$$

It is obvious from this equation that the quantum effects are important for small values of the scale factor and in the limit of the large scale factor can be neglected. Therefore, asymptotically the classical behavior is recovered. In this sense we can extract a repulsive force from the quantum potential (68) as

$$F_a = -\frac{\partial \mathcal{Q}}{\partial a} = \frac{1}{16}(3\alpha - 1)(3\alpha - 2)a^{-4}, \tag{69}$$

which may be interpreted as being responsible of the avoidance of singularity. For small values of a (near the big-bang singularity), this repulsive force takes a large magnitude and thus prevents the scale factor (and then the scalar field) to evolve to the classical singularity ($a \to 0, \phi \to \pm\infty$).

4 Conclusions

In this paper we have studied the classical and quantum dynamics of a scalar-metric cosmological model coupled to a perfect fluid in the context of the Schutz' representation. The use of the Schutz' formalism for perfect fluid allowed us to introduce the only remaining matter degree of freedom as a time parameter in the model. In terms of this time parameter, we have obtained the corresponding classical cosmology by evaluating the dynamical behavior of the cosmic scale factor and the scalar field. We have seen that the evolution of the universe based on the classical picture represents a late time power law expansion coming from a big-bang singularity. We then dealt with the quantization of the model in which we saw that the classical singular behavior will be modified. In the quantum model, we showed that the SWD equation can be separated and its eigenfunctions can be obtained in terms of analytical functions. By an appropriate superposition of the eigenfunctions, we constructed the corresponding wave packet. The wave function in this case shows a pattern in which there are two possible quantum states from which our present universe could have evolved and tunneled in the past from one state to another. The time evolution of this wave packet represents its motion along the larger a-direction while the scalar field ϕ remains almost constant. As time passes, our results indicated that the wave packets disperse and the minimum width being attained at $T = 0$, which means that the quantum effects are important for small enough T, corresponding to small a. The avoidance of classical singularities due to quantum effects, and the recovery of the classical dynamics of the universe are another important issues of our quantum presentation of the model. These questions have been investigated by two different methods. The time evolution of the expectation value of the dynamical variables and also their Bohmian counterparts have been evaluated in the spirit of the many-worlds and ontological interpretation of quantum cosmology respectively. We verified that a bouncing singularity-free universe is obtained in both cases. The use of the ontological interpretation has allowed us to understand the origin of the avoidance of singularity by a repulsive force due to the existence of the quantum potential. The repulsive nature of this force prevents the universe to reach the singularity.

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